Athemath Fall 2021 Admissions Quiz

Athemath Staff

Due September 1st, 2021

1 Instructions

For all of the problems below, **proof-based solutions are encouraged**. We would like you to explain all of your steps, instead of just giving an answer. LaTeX submissions and *neat*, dark handwriting submissions are both allowed.

We also encourage you to try the entire test. While later problems will generally be harder, they also play to different strengths and you may find one particularly easy.

Please do not use computer programs, Google, WolframAlpha, GeoGebra, etc. to help you find solutions. Additionally, please do not discuss this quiz with anyone else until after the application deadline has passed. **If you find the test difficult, that's because it's designed to be.** Only hard problems are worth doing, after all! If you get stuck, take a walk, try a different problem, or try a strategy you dismissed at first. And remember that you don't have to solve all—or even a majority—of the problems to get in.

Ask for clarifications by emailing Serena at serena.an@athemath.org. Submit your completed solutions to the application Google Form by **September 1st, 11:59PM Eastern**. As a reminder, only students of underrepresented genders can apply. Have fun!

2 The Problems!

Problem 1

Find the number of three-digit multiples of 6 such that no digit is zero.

Problem 2

All positive terminating decimals[1] are colored one of n colors such that no two decimals that differ in exactly one digit are the same color. What is the minimum possible value of n?

Problem 3

Find all unordered pairs of positive integers (a, b, c, d) such that $a^2 + b + c + d$, $b^2 + c + d + a$, $c^2 + d + a + b$, and $d^2 + a + b + c$ are all perfect squares.

Problem 4

For which positive integers $n \geq 3$ is it possible to place n nonnegative integers in a circle such that the n sums of two adjacent numbers are $1, 2, \ldots, n$ in some order?

Problem 5

Suppose scalene triangle ABC has incenter I. Let D be the foot from I to $\overline{BC}$, let M be the midpoint of $\overline{BC}$, and let N be the foot from M to line AI. If O is the circumcenter of triangle ADN, prove that $\overline{MO} \perp \overline{BC}$.

[1] A terminating decimal is a number that has a finite number of nonzero digits in its decimal representation. For example, 3.14 and 1 are terminating decimals, while π and $0.\overline{09}$ are not.