Athemath Winter 2021 Admissions Quiz

Athemath Staff

Due 11:59 PM PST, October 13

1 Instructions

For all the problems below, **proof-based answers are encouraged**. If you are not familiar with proofs, try to explain your thinking as best as you can. LaTeX submissions are encouraged, but *neat*, dark, cleanly-scanned handwriting is also permissible. **If your submission is not readable, we may not consider your application.**

We also encourage you to try the entire test. While later problems will generally be harder, they also play to different strengths and you may find one particularly easy. For multi-part problems, oftentimes the placement is solely determined by the difficulty of the last part, so you may find the first parts easier.

Please do not use computer programs, Google, WolframAlpha, GeoGebra, etc. to help you find solutions. In particular, please do not discuss this quiz with anyone else until after the application deadline has passed. If you need to cheat to get in, our program probably isn't a good fit for you. **And if you find the test difficult, that's because it's designed to be.** Only hard problems are worth doing, after all! If you get stuck, take a walk, try small cases, listen to some music, try a different problem, or try a strategy you dismissed at first. And remember that you don't have to solve all—or even a majority—of the problems to get in.

Ask for clarifications by emailing Andrew at vp@athemath.org. Submit your completed solutions to the student application form by **11:59 PM PST, October 13**. As a reminder, only students of underrepresented genders can apply. Have fun!

2 The Problems!

Problem 1

There exists some $k \neq 0$ such that quadratics $x^2 - 10x + k$ and $x^2 - 10x - k$ both have integer roots. Find $|k|$, and prove that your answer is unique.

Problem 2

Janabel has a piece of paper such that one side is red and the other side is blue. She cuts a triangle of area 126 out of the paper and labels its vertices A, B, and C. With the red side facing her, she folds point A over some line parallel to BC. It turns out that the total blue area she can see is twice the area of the total red area she can see! Find the total red area she can see.

Problem 3

Suppose $f(x)$ is a monic quadratic polynomial with distinct nonzero roots p and q, and suppose $g(x)$ is a monic quadratic polynomial with roots $p + \frac{1}{q}$ and $q + \frac{1}{p}$. If we are given that $g(-1) = 1$ and $f(0) \neq -1$, then there exists some real number r that must be a root of $f(x)$. Find r.

Problem 4

Find all pairs of three-digit integers $\overline{XYZ}$ and $\overline{ZYX}$ (with $X, Z \neq 0$) such that $\overline{XYZ} \cdot \overline{ZYX}$ is 1 more than a multiple of 101.

Problem 5

Let a "word" be any string of **distinct** English letters, and say that a word is n-pseudo-alphabetical if exactly n pairs of letters in the word are out of alphabetical order. For example, the word $ABECD$ is 2-pseudo-alphabetical, because the pairs of letters (E, C) and (E, D) are not in alphabetical order, but every other pair of letters is.

1. Find, in terms of k, the number of 1-pseudo-alphabetical words that can be formed from the first k letters in the alphabet.

2. Find, in terms of k, the number of 2-pseudo-alphabetical words that can be formed from the first k letters in the alphabet.